where $\delta^{r(i)}$ is a 3×1 matrix whose elements are the Kronecker delta and

$$\delta^{r(i)} = [\delta_1^{r(i)}, \delta_2^{r(i)}, \delta_3^{r(i)}]^T$$

Based on Eqs. (2) and (14), the angular velocity relation could be also computed on digital computer for any sequence of rotations, but as is pointed out previously, this simple algorithm includes unnecessary multiplications and additions. Such unnecessary calculations are shown to be avoided with the following simple algorithm.[6]

First, compute the indices, n_1, n_2, and n_3 (Eqs. (10)) and the following

$$k_1 = \mathrm{MOD}(r(2) - r(1) + 3, 3) \tag{15a}$$

$$k_2 = \mathrm{MOD}(r(3) - r(2) + 3, 3) \tag{15b}$$

$$k_3 = \mathrm{MOD}(k_2, 2) + 1 \tag{15c}$$

and

$$j_1 = 1 \quad \text{and} \quad j_2 = 2 \quad \text{if} \quad k_1 \neq k_2 \tag{16a}$$

$$j_1 = 2 \quad \text{and} \quad j_2 = 1 \quad \text{if} \quad k_1 = k_2 \tag{16b}$$

In addition to these, the following sign functions are defined:

$$p_1 = (-1)^{k_1+1}, \quad p_2 = (-1)^{k_2+1}, \quad p_3 = (-1)^{k_2} \tag{17}$$

Then, compute auxiliary functions $u(\bullet)$, $v(\bullet)$, etc., by

$$u(j_1) = \cos\phi_2 \tag{18a}$$

$$u(j_2) = p_1 \sin\phi_2 \tag{18b}$$

$$v(k_2) = \cos\phi_3 \tag{18c}$$

$$v(k_3) = \sin\phi_3 \tag{18d}$$

and

$$v_3 = p_3 v(2) \tag{18e}$$

$$v_4 = p_2 v(1) \tag{18f}$$

With this preparation, the elements of ω, designated by $\omega(i)$, $i = 1,2,3$ (which refer to the axis number) are given in terms of $\dot{\phi}_i$ $(i = 1,2,3)$

$$u_2 = u(2)\dot{\phi}_1 \tag{19a}$$

$$\omega(n_1) = u(1)\dot{\phi}_1 + \dot{\phi}_3 \tag{19b}$$

$$\omega(n_2) = v(1)u_2 + v(2)\dot{\phi}_2 \tag{19c}$$

$$\omega(n_3) = v_3 u_2 + v_4 \dot{\phi}_2 \tag{19d}$$

Conversely, $\dot{\phi}_i$ $(i = 1,2,3)$ are given in terms of ω

$$\dot{\phi}_1 = v(1)\omega(n_2) + v_3\omega(n_3)/u(2) \tag{20a}$$

$$\dot{\phi}_2 = v(2)\omega(n_2) + v_4\omega(n_3) \tag{20b}$$

$$\dot{\phi}_3 = \omega(n_1) - u(1)\dot{\phi}_1 \tag{20c}$$

As previously, the computations of Eqs. (15)-(17) are required only once for a given order of sequence, so that they are completely omitted as long as the $r(i)$ remain unchanged. Once these indices are computed, the computations of Eqs. (18) and (19) are to be computed for the case of $\dot{\phi} \rightarrow \omega$, or those of Eqs. (18) and (20) are for the case of $\omega \rightarrow \dot{\phi}$.

Conclusion

Algorithms have been established for computation of kinematical relations for three attitude angle systems used in digital computer simulation of spacecraft attitude dynamics. For this establishment, a general representation of a planar rotation is introduced (Eqs. (2)) and the transformation is accomplished as shown in Eqs. (7)-(13).

The established algorithms possess inversion capability, that is, with an appropriate indication of inversion the transformation from body to reference frame is computed in the transformation program from reference to body frame. In addition, computation of the time derivatives of three attitude angles from body rates, and the converse are carried out in the angular velocity computation algorithm [Eqs. (15)-(20)].

Both programs require as input the order of the rotation sequence and three attitude angles only, so that laborious manual calculation of matrices is completely avoided. Since particular care was paid to minimize the number of multiplications, there is no significant penalty on computation time in spite of the generality and utility of the programs.

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Boundary-Layer Development on Moving Walls Using an Integral Theory

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I. Introduction

THE aim of this work was to develop a technique which might be applied to compute the boundary-layer growth on moving walls. This method was applied to investigate the forces acting on a spinning cylinder in crossflow.[1]

The method presented here involves integrating the coupled integral-momentum and integral-energy equations for the dependent variables; a shape factor K and the momentum thickness in terms of the independent variable x, the distance along the wall. Other shape factors appearing in the integral equations are related to K and u_w/u_e (wall velocity to boun-

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dary-layer edge velocity ratio) by assuming that moving-wall similarity solutions can provide a family of velocity profiles which approximate those of the nonsimilar boundary layer. These similarity solutions are characterized by a single shape factor and u_w/u_e; they have been found for a wide range of u_w/u_e and pressure-gradient parameter values.[2]

This particular integral method was chosen because similar methods[3,4] have been used to solve problems involving regions of flow somewhat analogous to the upstream-moving wall region on a rotating cylinder. Here the boundary-layer development is determined mainly by local overall momentum and energy balances with stability considerations establishing the regions of the boundary layer that cannot be calculated because of large amounts of reverse flow.

Boundary layers calculated by the integral techniques are compared with an exact solution of Rott[5] and an approximate solution by Glauert[6] which becomes exact in the limit as the wall velocity goes to zero. With the present approach, boundary layers may be calculated for both upstream and downstream moving walls. Other investigators[7,8] using finite-difference methods, also have been able to calculate boundary layers with reverse flow by specifying the displacement thickness which, in turn, determines the pressure distribution. Catherall and Mangler[7] solved a boundary-layer flow which included the separation point and the region downstream where the maximum reverse-flow velocity to external-flow velocity ratio is near -0.05. More recently, Carter and Wornom[8] computed, in an approximate manner, boundary-layer characteristics where the ratio of the velocities was near -0.1. With the present integral technique, however, boundary layers with velocity ratios up to -0.3 may be calculated.

II. Integral-Momentum and Integral-Energy Equations

The Prandtl equations for steady constant-property incompressible two-dimensional boundary-layer flow are

$$\bar{u}\frac{\partial \bar{u}}{\partial \bar{x}} + \bar{v}\frac{\partial \bar{u}}{\partial \bar{y}} = \bar{u}_e \frac{d\bar{u}_e}{d\bar{x}} + \nu \frac{\partial^2 \bar{u}}{\partial \bar{y}^2} \tag{1}$$

$$\frac{\partial \bar{u}}{\partial \bar{x}} + \frac{\partial \bar{v}}{\partial \bar{y}} = 0 \tag{2}$$

With boundary conditions

$$\bar{u}(\bar{x},0) = \bar{u}_w, \quad \bar{v}(\bar{x},0) = 0, \quad \bar{u}(\bar{x},\bar{y}) \rightarrow \bar{u}_e \text{ as } \bar{y} \rightarrow \infty \tag{3}$$

Here $\bar{x}$ is the physical coordinate along the surface, $\bar{u}$ is the physical velocity component in the $\bar{x}$ direction, $\bar{y}$ is the normal physical coordinate, and $\bar{v}$ is the physical velocity component in the $\bar{y}$ direction. The subscript e denotes conditions at the outer edge of the boundary layer and subscript w denotes conditions at the wall. The kinematic viscosity is ν.

The radius a of the cylinder will be taken as a fundamental length. Using this fundamental length, nondimensionalized independent coordinates are defined as $x \equiv \bar{x}/a$ and $y \equiv \bar{y} Re_d^{1/2}/a$, where $Re_d = 2u_0 a/\nu$. All velocities are nondimensionalized with respect to u_0, the freestream velocity.

The corresponding integral momentum equation can be written as

$$d\theta^2/dx = 2[2T - (2+H)\theta^2(du_e/dx)]/u_e \tag{4}$$

Here θ is a nondimensional momentum thickness:

$$\theta = \left(\frac{1}{u_e^2}\right)\int_0^\infty u(u_e - u)\,dy$$

The shape factors H and T are defined as in Rosenhead.[9]

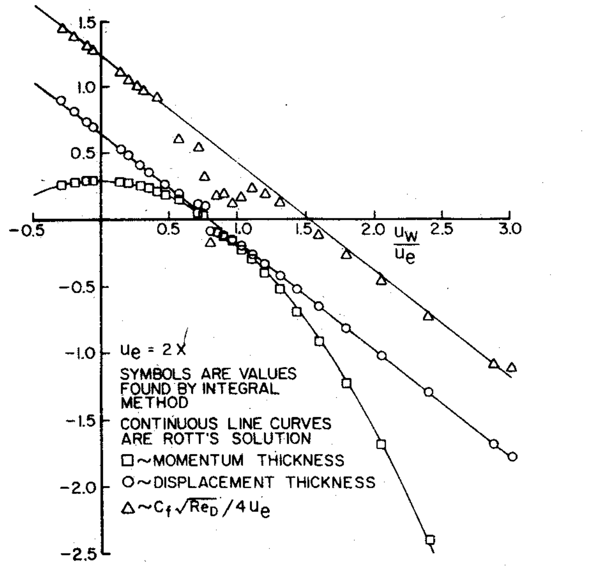

Fig. 1 Integral method comparison with Rott's solution; line curves are from Rott's solution.

The integral energy equation is

$$d(K^2\theta^2)/dx = 2[4K(L + u_w T/u_e) - 3K^2\theta^2(du_e/dx)]/u_e \tag{5}$$

The shape factors K and L also are defined as in Rosenhead.

The two differential equations, Eqs. (4) and (5), are coupled and can be solved simultaneously for the dependent variables K and θ^2, as a function of x with suitable initial conditions. The shape factors H, T, and L can be considered as functions of K and u_w/u_e through the assumption that the nonsimilar shape-factor relationships can be approximated by the moving-wall similarity solutions.

Initial values of θ and K must be defined in order to start the integration of Eqs. (4) and (5). At the stagnation point on a cylindrical surface, θ becomes infinite as defined here but the integration can be started in the stagnation region using Rott's[5] stagnation region solution.

III. Comparison with Moving-Wall Solutions

A comparison of the method can be made with Rott's[5] solution. The nonsimilar boundary-layer velocity profiles obtained by Rott[5] correspond to the only exact solution for a nonaccelerating moving wall in two-dimensional steady flow. The solution obtained by Rott[5] corresponds to the problem of a flow whose direction is perpendicular and towards a flat plate with its surface moving at constant velocity.

The comparison with Rott's[5] solution is made in Fig. 1. The boundary layer was computed for the range $-0.3 \leq u_w/u_e \leq 3.0$. Stable accurate integration of the boundary layer could not be performed for $u_w/u_e \leq -0.3$. The agreement is quite good except in a region centered near $u_w/u_e = 0.8$. In the upstream part of this region, the boundary-layer profiles for Rott's solution have supervelocities, i.e., velocities greater than either the wall or external-flow velocities exist in the boundary-layer flow. Libby and Liu's[10] similarity supervelocity profiles generalized to the moving wall can initially be used in this region with the integral technique. However, as Rott's profiles change from supervelocity flows to flows without supervelocities, Libby and Liu's[10] profiles do not possess maximum flow velocities near that of the velocities for the outer boundary conditions. To compute in this region, interpolation was performed between the integral values for the similarity profiles without supervelocities and the similarity profiles with supervelocities. Apparently the Libby and Liu[10] profiles and the conventional similarity profiles

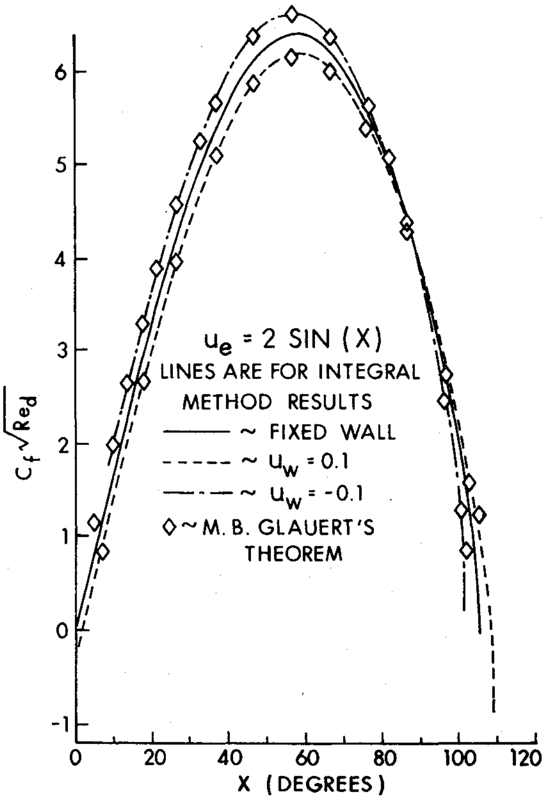

Fig. 2 Comparison with Glauert's result for a cylinder.

were so different from each other that interpolating between their integral values did not yield good results. As integration proceeds to smaller values of u_w/u_e (downstream of $u_w = u_e$) the momentum and displacement thickness quickly converge back to the correct level with the skin friction being somewhat slower to converge. Thus, the integral boundary-layer calculation of Rott's solution is shown to be stable, so that after the interval of inaccurate calculation, the integral quantities converge rapidly back to Rott's[5] solution. For the application to rotating cylinders, the region of inaccuracy did not affect the results significantly. However, for some other possible applications, the similarity profiles might be augmented in the region of less accuracy by some suitably designed profiles satisfying some compatibility conditions.

A comparison of the method can also be made with results obtained by a theorem of Glauert.[6] This theorem can be used to estimate moving-wall boundary-layer skin friction for the case of a slowly rotating cylinder. The result of the theorem applied to rotating cylinders is the following: if one knows the pressure distribution on a rotating cylinder and the boundary-layer development on the equivalent fixed cylinder, the boundary-layer development on the slowly rotating cylinder may be calculated approximately. For the slowly rotating cylinder a result obtained by Glauert[6] is that

$$\tau \simeq [\tau_l^2 - 2\tilde{u}_w \mu \rho \tilde{u}_e (d\tilde{u}_e/d\tilde{x})]^{1/2} \tag{6}$$

where τ is the shear stress for the moving wall and τ_l is the shear stress for the fixed wall. Here the subscript 1 denotes the fixed wall conditions. Equation (6) can be nondimensionalized for this particular problem with the result:

$$c_f(Re_d)^{1/2} \simeq [(c_f)_l^2 Re_d - 32u_w \sin(2x)]^{1/2} \tag{7}$$

The skin friction results obtained by the integral method for cylinder rotation rates of -0.1, 0.0, and 0.1 are shown in Fig. 2 as a function of position on the cylinder. The skin friction results using Glauert's theorem are included for comparison. The nondimensional velocity distribution is 2 sin$x$. The results obtained by Glauert's[6] theorem for slowly rotating cylinders agree well with the results obtained by the integral technique except near separation where Glauert's approximation tends to break down.

IV. Conclusions

A two-equation integral technique for calculating the incompressible two-dimensional boundary-layer development on a moving wall has been developed. By comparison with Rott's[5] solution and a result obtained by Glauert's[6] theorem, it has been determined that accurate calculations of boundary-layer development can be obtained over the major portion of a slowly rotating cylinder. For the upstream moving wall, the lower limit for accurate stable integration of the boundary is found to be $u_w/u_e = -0.3$ and a lower limit for u_w/u_e is to be expected from the parabolic nature of the boundary-layer equations. It appears that with the integral technique, larger reverse flows can be handled than with finite-difference methods.[7,8] The flows used with the two techniques were different, however, and so the lack of a direct comparison prevents complete confirmation.

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Rapid Analysis of Damaged Structure

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An iteration algorithm based upon known mathematical principles is presented for comment and exploitation. The matrix operations and decisions can be performed very rapidly in any digital computer.

Digital procedures for the analysis of a loaded structure are readily available. Unfortunately, the programs require the solution of large numbers of simultaneous equations which

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